Chapter 2 — Forces in Electromagnetic Launchers

Phil Putman
July 2006

The forces in all electric machines, including the accelerating forces in electromagnetic launchers, arise because of the relative motion of charged particles. This force is called the Lorentz force, after the Dutch physicist Hendrik Lorentz. The force is a consequence of a transformation between coordinate systems, as presented below. Understanding this derivation is not a prerequisite for launcher design, but it does help in appreciating the fundamental nature of the connection between electromagnetism and mechanics. The result of the derivation is just the equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

where q is the charge of the particle in question, $\mathbf{E}$ is the electric field, $\mathbf{u}$ is the velocity of the particle, and $\mathbf{B}$ is the magnetic field. If the charge q is an electron traveling in a wire, the force is transmitted from the electron to the wire electrostatically. When the negatively charged electron is displaced, it pulls the positively charged ions in the lattice along with it.

In order to calculate the performance of a launcher, it is necessary to repeat a cycle of finding motion due to the forces, then the effect of motion on the currents, which in turn produce different forces, etc. A section on induced currents is also included below.

Origin of the Lorentz force

This derivation for the force between two particles or circuits when one is moving relative to the other follows [Shadowitz]. Taking one circuit to be stationary, while the other moves with a speed u in the $+x$ direction, gives coordinate systems as shown in Fig. 2-1. The moving coordinate frame will be referred to as primed, while the

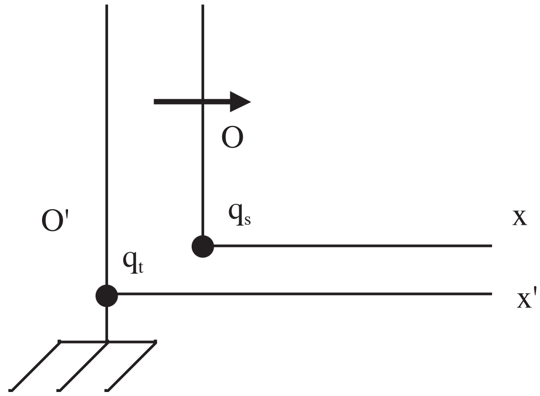

Figure 2-1

stationary frame is unprimed. The time and position transformations between reference frames are

$$x = \frac{x' + \beta ct'}{\sqrt{1-\beta^2}}$$

$$ct = \frac{\beta x' + ct'}{\sqrt{1-\beta^2}}.$$

These equations are called the Lorentz transformation, and can be used to express E' and B' in terms of E and B (regardless of the source of the fields),

$$E'_x = E_x$$

$$E'_y = \frac{E_y + uB_z}{\sqrt{1-\beta^2}}$$

$$E'_z = \frac{Ez_y - uB_y}{\sqrt{1-\beta^2}},$$

$$B'_x = B_x$$

$$B'_y = \frac{B_y - \frac{u}{c^2}E_z}{\sqrt{1-\beta^2}}$$

$$B'_z = \frac{B_z + \frac{u}{c^2} E_y}{\sqrt{1-\beta^2}}.$$

The force on a test charge q_t, due to a source charge q_s, as measured by observer O', can be expressed in the primed coordinate frame as

$$F'_{Ox} = q_t E_x$$

$$F'_{Oy} = q_t E_y$$

$$F'_{Oz} = q_t E'_z.$$

The force transformation equations for the force on q_t as measured by O are

$$F_x = F'_{Ox}$$

$$F_y = \sqrt{1-\beta^2} F'_{Oy}$$

$$F_z = \sqrt{1-\beta^2} F'_{Oz}$$

Combining the force equations with the expressions for E' and B' produces

$$F_x = q_t E_x$$

$$F_y = q_t (E_y + uB_z)$$

$$F_z = q_t (E_z - uB_y),$$

or, in vector notation,

$$\mathbf{F} = q_t (\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

This is the Lorentz force equation, obtained by transformation from a proper to a nonproper observer.

The same approach can be taken to finding the effect of relative motion on the currents in each part of the circuit. For fields $B = a_z B$, E = 0, relative to an observer *O*, if *O′* moves toward +*x* at speed *u*, the fields appear to *O*' as

$$E'_x = 0$$

$$E'_y = \frac{0 - uB}{\sqrt{1-\beta^2}} \approx -uB$$

$$E'_z = \frac{0 + 0}{\sqrt{1-\beta^2}} = 0$$

$$B'_x = 0$$

$$B'_y = \frac{0 - 0}{\sqrt{1-\beta^2}} = 0$$

$$B'_z = \frac{B + 0}{\sqrt{1-\beta^2}} \approx B$$

.

For non-relativistic speeds, the moving observer experiences the same field as the stationary observer, and, in addition, experiences an electric field orthogonal to *u* and *B*. Integrating this electric field produces the voltage in the generator equation, $V=l\mathbf{u}\times\mathbf{B}$.

Equivalent representations

It is often convenient to calculate forces using an equation expressed in terms of the mutual inductance between circuit elements,

$$F = -I_1 I_2 \nabla M_{12},$$

in which I_1 and I_2 are the currents in the two circuit elements, and M_{12} is their mutual inductance. The result is the same as that found using the Lorentz force equation directly. Other equivalent methods include formulas for the force between current filaments, magnetic pressure, and the Maxwell stress tensor.

Induced currents

As shown above, force and induced voltage can both arise from transformation between reference frames. A changing magnetic field also induces a voltage, which can be described by Faraday's law, in point form given by

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}.$$

An equivalent representation is

$$V = \frac{-d\Phi}{dt}.$$

The rate of change in Φ with time can be caused either by a changing current in another circuit, or circuit motion. The components due to motion and changing field can be derived making use of the circuit shown in Fig. 2-2. The figure shows a circuit at position *C* at time *t*, then at a later

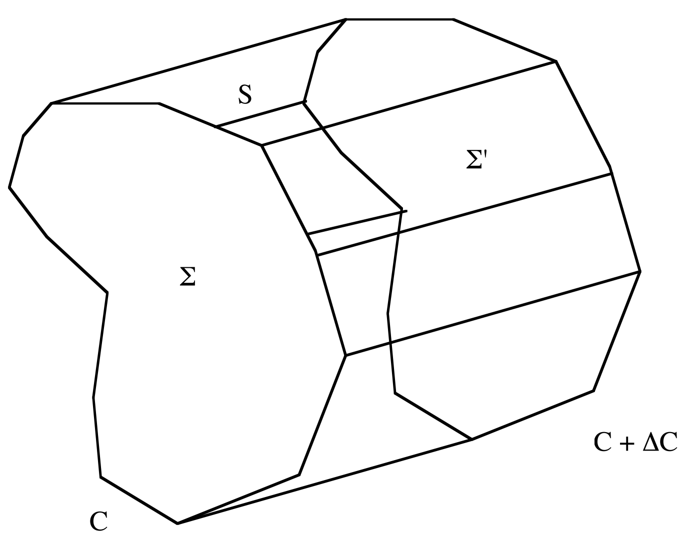

Figure 2-2

time $t+\Delta t$ at position $C+\Delta C$. The field is nonuniform. The area swept out by the circuit is S. Expressing voltage in terms of change in flux gives the equation

$$V = \frac{-d\Phi}{dt} = -\lim_{\Delta t \to 0}[\frac{\Delta\Phi}{\Delta t}] = -\lim_{\Delta t \to 0}[\frac{1}{\Delta t}(\int_{\Sigma'} B \bullet ds - \int_{\Sigma} B \bullet ds)]$$

Expressing the flux at time $t+\Delta t$ as the flux at time t plus the gain in flux during Δt plus the gain due to the displacement $u\Delta t$ produces

$$V = -\lim_{\Delta t \to 0}[\frac{1}{\Delta t}((\int_{\Sigma} B \bullet ds + \int_{\Sigma'} (\frac{\delta}{\delta t} B\Delta t) \bullet ds + \int_{C} B \bullet (u\Delta t \times dr)) - \int_{\Sigma} B \bullet ds)]$$

Rearranging,

$$V = -\frac{\delta}{\delta t}\int_{\Sigma} B \bullet ds + \int_{C} (u \times B) \bullet dr$$

The induced voltage due to motion is the same as the one derived using the Lorentz transformation, and the induced voltage due to the changing field is given by the derivative of the change in flux that is due to the nonuniform field.

References

A. Shadowitz. *The Electromagnetic Field*, Dover Publications , Inc., New York, 1975.